Differential Geometry-II Final Test (BMath-2022)

Instructions: Total time 3 Hours. Solve any FIVE problems, for a maximum score of 50. Use terminologies, notations and results as covered in the course, no need to prove such results. If you wish to use a problem given in an assignment, homework or quiz, please supply its solution.

1. Let $G \subset GL_n(\mathbb{R})$ be a Lie subgroup. Let $\mu : G \times G \to G$ denote the multiplication map and $\iota : G \to G$ be the inverse map $\iota(x) = x^{-1}$. Show that $D\mu(e,e)(X,Y) = X + Y$ and $D\iota(e)(X) = -X$ for all $X, Y \in \mathrm{Lie}(G)$. Here e is the identity element of G. (6+4)

2. Let V be a finite dimensional vector space over $\mathbb{R}$ and $B : V \times V \to \mathbb{R}$ be a nondegenerate bilinear form on V. Let
$$G = \{g \in GL(V) | B(v,w) = B(g(v), g(w)) \text{ for all } v, w \in V\}.$$
Prove that G is a closed Lie subgroup of $GL(V)$ and
$$\mathrm{Lie}(G) = \{X \in \mathrm{End}(V) | B(X(v), w) + B(v, X(w)) = 0 \text{ for all } v, w \in V\}.$$
(4+6)

3. Let M, N be smooth manfolds and $f : M \to N$ be a smooth map. Let $g \in C^\infty(N)$. Prove that $\mathrm{d}(g \circ f) = f^*(\mathrm{d}g)$, where $f^*(\omega)$ denotes the pullback of a differential form ω on N, by f. (10)

4. Let $n \geq 1$ be an integer. Construct a nowhere vanishing differential 1-form on the sphere $S^{2n-1} \subset \mathbb{R}^{2n}$. (10)

5. Prove that any Lie group G is orientable, i.e. there exists a nowhere vanishing top degree differential form on G. (10)

6. Let M be a smooth manifold, $N \subset M$ a submanifold and $i : N \to M$ the inclusion map. Is it true that $i^* : D^r(M) \to D^r(N)$ is injective? Justify your answer. Here $D^r(M)$ denotes the space of degree r-differential forms on M. (10)